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Reflection and refraction of a transient temperature field at a plane interface using Cagniard–de Hoop approach

Margarita L. Shendeleva
School of EEIE, South Bank University, 103 Borough Road, London SE1 0AA, United Kingdom
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An instantaneous line heat source located in the medium consisting of two half-spaces with different thermal properties is considered. Green's functions for the temperature field are derived using the Laplace and Fourier transforms in time and space and their inverting by the Cagniard–de Hoop technique known in elastodynamics. The characteristic feature of the proposed approach consists in the application of the Cagniard–de Hoop method to the transient heat conduction problem. The idea is suggested by the fact that the Laplace transform in time reduces the heat conduction equation to a Helmholtz equation, as for the wave propagation. Derived solutions exhibit some wave properties. First, the temperature field is decomposed into the source field and the reflected field in one half-space and the transmitted field in the other. Second, the laws of reflection and refraction can be deduced for the rays of the temperature field. In this connection the ray concept is briefly discussed. It is shown that the rays, introduced in such a way that they are consistent with Snell's law do not represent the directions of heat flux in the medium. Numerical computations of the temperature field as well as diagrams of rays and streamlines of the temperature field are presented.

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I. INTRODUCTION

It is well known that temperature fields generated by time-periodic sources exhibit wavelike features, such as the ability to reflect or scatter at boundaries. Wave properties of temperature fields have been studied intensively during the last two decades and have been utilized in various techniques for obtaining thermal wave images and measuring thermal properties of matter. Nevertheless, our understanding of thermal waves is still incomplete. In particular, this concerns the behavior of thermal waves encountering an interface between two different media. Extending the analogy with conventional waves, one can expect to observe something like Snell's law of refraction and other associated phenomena. Fortunately, modern experimental techniques make such observations possible and recently an experimental verification of Snell's law for plane thermal waves was reported [1]. If Snell's law holds, the questions about the total reflection and head waves arise, which cannot be resolved within existing models. This indicates the necessity of further experiments and theoretical modeling.

While the applicability of wave analogies to harmonic thermal waves does not cause objections, the temperature fields generated by impulsive heat sources are usually considered to be purely diffusional. However, the absence of periodic component in transient temperature field does not prevent it from propagation and the presence of boundaries or other restrictions can change the temperature distribution dramatically. Taking into account the fact that the transient solution can be expressed as a sum of harmonic components by means of the Fourier integral, one can expect to observe the reflection and refraction of the transient temperature fields at the interface analogous to frequency domain and this does not contradict their diffusive nature. Nevertheless, the laws of reflection and refraction for diffusive waves may be quite different from those for hyperbolic waves.

Green's function technique seems to be the most adequate for studying the temperature fields from impulsive point sources. For simple cases, such as regions bounded with a vacuum, Green functions for point heat sources were found by method of images both in time [2] and frequency domains [3,4]. For two different media in contact the method of images permits considering only one-dimentional case. A solution for three-dimensional (3D) point, instantaneous heat source in the medium with an interface was obtained by means of integral transformations [2,5], but it is difficult for physical interpretation. Our research does not pretend to replace the above-mentioned solution, because we will focus on 2D case. The approach proposed in this paper is based on the method devised by Cagniard in elastodynamics.

Cagniard [6] developed his method in 1930s when studying propagation of seismic waves. He considered reflection and refraction of spherical compressional elastic wave initiated by a point source in the two homogeneous elastic semi-infinite media separated by a plane interface. Later, de Hoop [7] showed that much simplification could be achieved for the two-dimensional case. The general outline of the method consists in the application of the Laplace transform with respect to time and the Fourier or Hankel transforms with respect to the spatial variable parallel to the interface. After satisfying boundary conditions the inversions of transforms are performed by modification of the contour of integration in the transformed domain of the spatial variable. Nowadays this method is used in seismology, laser ultrasonics, cracks dynamics, electromagnetic wave propagation, etc.

There were several reports about application of the Cagniard–de Hoop method to diffusive problems. First is de Hoop's research on transient diffusive electromagnetic fields in conductive media [8]. Second is the paper of Oliver [9] on pressure transfer in porous media. The mathematical statement of the latter problem is very similar to ours, but Oliver first solves an auxiliary wave equation and then transforms its solution to the solution of the diffusive problem by using

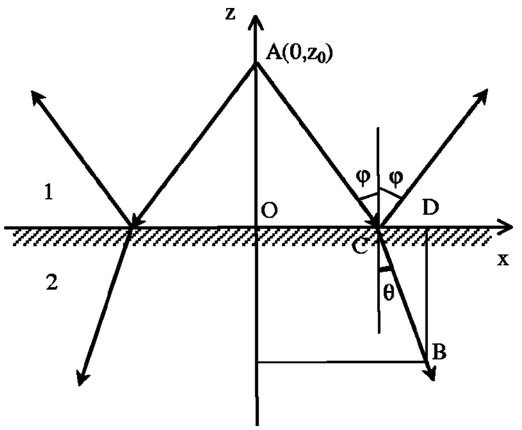

FIG. 1. Geometry of the model. A pulsed line heat source is located at the point A of the medium consisting of two half-spaces (1 and 2) with different thermal properties. φ denotes the angle of incidence and the angle of reflection, and θ denotes the angle of refraction for the rays of temperature field.

a special integral transform. The derivation proposed by us is straightforward and more transparent.

In this paper, a temperature field generated by an instantaneous line heat source in the medium consisting of two half-spaces with different thermal properties is considered. A system of heat conduction equations is solved using Laplace transform with respect to time and Fourier transform with respect to the spatial variable running along an interface. The modification of contour in the Fourier domain according to the Cagniard–de Hoop procedure allows us to evaluate an integral with respect to the Laplace variable and considerably simplify the remaining integration. An analysis of analytical solutions obtained for the reflected and transmitted temperature fields and examples of numerical computations are given.

II. DERIVATION OF TEMPERATURE GREEN FUNCTIONS USING CAGNIARD–DE HOOP TECHNIQUE

A. Application of integral transforms

Consider two semi-infinite media in perfect thermal contact at a plane interface with an instantaneous line heat source located in one of them, as shown in Fig. 1. Heat diffusion is governed by the system of two heat conduction equations

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)T_1-\frac{1}{D_1}\frac{\partial T_1}{\partial t}=-\frac{1}{D_1}\delta(t)\,\delta(x)\,\delta(z-z_0), \tag{1}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)T_2-\frac{1}{D_2}\frac{\partial T_2}{\partial t}=0, \tag{2}$$

subjected to boundary conditions at $z=0$

$$T_1=T_2,\quad k_1\frac{\partial T_1}{\partial z}=k_2\frac{\partial T_2}{\partial z}, \tag{3}$$

and zero initial condition.

In these equations, T_j is the temperature rise above the equilibrium state, D_j is the thermal diffusivity, k_j is the thermal conductivity of the each medium, indices $j=1,2$ are referred to the upper $(z>0)$ and lower $(z<0)$ half-spaces, respectively.

It should be noted that the heat source in the equation (1) is considered in the form $q=q_0\delta(t)\,\delta(x)\,\delta(z-z_0)$, where q is the power per unit volume (W/m^3), q_0 is a factor of dimension Jm^{-1}, δ denotes Dirac delta function. In Eq. (1) we assume $q_0/(\rho_1c_1)=1$ Km^2, where ρ_1 is the density of matter in the upper half-space and c_1 is the specific heat. Such a choice allows us to keep the correct dimension for temperature (K) and provides convenient normalization for the temperature field in the infinite homogeneous space: $\int\int T\,dx\,dz=1$, where integration is taken in infinite limits. For arbitrary q_0 all solutions have to be multiplied by $q_0/(\rho_1c_1)$.

To find a solution to the problem the Laplace transform with respect to time variable

$$u_j=\int_0^\infty e^{-pt}T_j\,dt \tag{4}$$

and the Fourier transform with respect to spatial variable x

$$U_j=\int_{-\infty}^{\infty}u_je^{-i\xi x}\,dx \tag{5}$$

are applied. Then the solutions for the Laplace transforms, which satisfy boundary conditions, can be found in the form

$$u_1=u_S+u_R, \tag{6}$$

$$u_2=u_P, \tag{7}$$

where u_S is the Laplace transform of the source field

$$u_S=\frac{1}{4\pi D_1}\int_{-\infty}^{\infty}\exp\left[\sqrt{\frac{p}{D_1}}\left(i\xi x-|z-z_0|\sqrt{\xi^2+1}\right)\right]\frac{d\xi}{\sqrt{\xi^2+1}}, \tag{8}$$

and u_R and u_P are the Laplace transforms of the reflected and transmitted field, respectively,

$$u_R=\frac{1}{4\pi D_1}\int_{-\infty}^{\infty}\exp\left[\sqrt{\frac{p}{D_1}}\left(i\xi x-(z+z_0)\sqrt{\xi^2+1}\right)\right]\frac{R(\xi)d\xi}{\sqrt{\xi^2+1}}, \tag{9}$$

$$u_P=\frac{1}{4\pi D_1}\int_{-\infty}^{\infty}\exp\left[\sqrt{\frac{p}{D_1}}\left(i\xi x+z\sqrt{\xi^2+n^2}-z_0\sqrt{\xi^2+1}\right)\right]\times\frac{P(\xi)d\xi}{\sqrt{\xi^2+n^2}}. \tag{10}$$

During the derivation of Eqs. (8)–(10) the new variable $\xi'=\xi\sqrt{D_1/p}$ was introduced, then the prime was omitted.

Expressions (9) and (10) contain functions $R(x)$ and $P(x)$, which can be called the reflection and transmission coefficients for the Laplace transforms,

$$R(\xi)=\frac{\sqrt{\xi^2+1}-\chi\sqrt{\xi^2+n^2}}{\sqrt{\xi^2+1}+\chi\sqrt{\xi^2+n^2}}, \tag{11}$$

$$P(\xi)=\frac{2\sqrt{\xi^2+n^2}}{\sqrt{\xi^2+1}+\chi\sqrt{\xi^2+n^2}}, \tag{12}$$

where two dimensionless parameters n and χ are defined by the ratios of the thermal properties of the media

$$n=\sqrt{\frac{D_1}{D_2}}, \quad \chi=\frac{k_2}{k_1}. \tag{13}$$

The next step is to find the inversions for the Laplace transforms

$$T_j=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} u_j e^{pt}\,dp. \tag{14}$$

For the source field u_S it can be done by means of elementary manipulation, giving a Green function of point heat source in the infinite space, as expected:

$$T_S=\frac{1}{4\pi D_1 t}\exp\left[-\frac{x^2+(z-z_0)^2}{4D_1 t}\right]. \tag{15}$$

The inversion for the reflected u_R and transmitted u_P fields is performed using the Cagniard–de Hoop technique as described in the following sections.

B. Reflected field

Consider the inversion of the Laplace transform for the reflected field

$$T_R=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} u_R e^{pt}\,dp, \tag{16}$$

where u_R is defined by Eq. (9). The essence of the Cagniard approach to inversion consists in modification of the integration contour in the domain of transformed variable ξ, which is now considered as a complex variable.

For the reflected field the Cagniard contour in the complex ξ plane can be introduced as

$$-i\xi x+(z+z_0)\sqrt{\xi^2+1}=r\sqrt{\beta^2+1}, \tag{17}$$

where β is the real parameter. When parameter β changes from zero to infinity, such defined contour represents a hyperbola in the ξ plane consisting of two parts ξ_- and ξ_+, as shown in Fig. 2(a),

$$\xi_{\pm}=i\sqrt{\beta^2+1}\,\sin\varphi\pm\beta\cos\varphi. \tag{18}$$

Here the polar coordinates r and φ are introduced, with φ being the angle of incidence, as shown in Fig. 1,

$$r^2=x^2+(z+z_0)^2, \tag{19}$$

$$\sin\varphi=x/r. \tag{20}$$

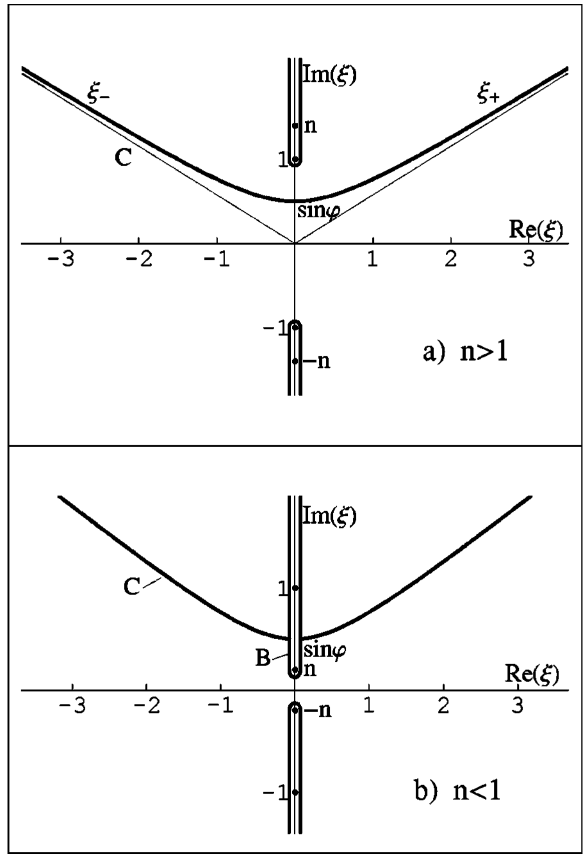

FIG. 2. (a) For $n>1$ the contour of integration is the Cagniard hyperbola (C); (b) for $n<1$ and $\sin\varphi>n$ the contour of integration consists of the Cagniard hyperbola (C) and the bypass (B) of the branch point $\xi=in$.

The hyperbola intersects the imaginary axis at the point $\sin\varphi$ when $\beta=0$.

The integrand in Eq. (9) has two pairs of branch points $\xi=\pm i$ and $\xi=\pm in$, which correspond to the zeros of radicals $\sqrt{\xi^2+1}$ and $\sqrt{\xi^2+n^2}$. For further consideration it is important to distinguish between two cases, $n>1$ and $n<1$.

For $n>1$, which is the case when the heat source is located in the medium with the higher thermal diffusivity, the branch cuts start from the points ± 1 on the imaginary axis as shown in Fig. 2(a) and the Cagniard contour does not cross the branch cut.

The region between the Cagniard contour and the real ξ axis does not contain singularities and the integration along the real axis can be replaced by the integration along the Cagniard contour according to the Cauchy theorem

$$T_R=\frac{1}{2\pi i}\,\frac{1}{4\pi D_1}\int_{-i\infty}^{+i\infty}\int_C e^{pt} \times\exp\left[-r\sqrt{\frac{p(1+\beta^2)}{D_1}}\right]\frac{R(\xi)}{\sqrt{\xi^2+1}}\,d\xi\,dp. \tag{21}$$

In the internal integral C denotes the integration along the Cagniard contour. Evaluating the integral with respect to the

Laplace variable and taking into account the symmetry of the Cagniard hyperbola relative to the imaginary ξ axis

$$\xi_- = -(\xi_+)^* \tag{22}$$

we obtain the Green function corresponding the reflected temperature field for $n>1$

$$T_R = \frac{r}{4(\pi D_1 t)^{3/2}} \int_0^\infty \exp\left[-\frac{(\beta^2+1)r^2}{4D_1 t}\right] \mathrm{Re}[R(\xi)] d\beta. \tag{23}$$

Function $R(x)$ is defined by Eq. (11) and any brunch $\xi_\pm$ can be taken.

The reflected field (23) has a character of waves emanating from the image source located at the point with coordinates $(0,-z_0)$. Thus we can conclude that the angle of reflectance is equal to the angle of incidence for rays of temperature field, as shown in Fig. 1.

When the heat source is located in the medium with the lower thermal diffusivity $n<1$ and the brunch cuts start from the points $\pm n$ on the imaginary axis, as shown in Fig. 2(b). In this situation, if the angle of incidence satisfies the condition $\sin\varphi<n$, the Cagniard hyperbola does not cross the brunch cut and all above considerations and the formula (23) are valid. If $\sin\varphi>n$, the brunch point $+n$ has to be bypassed along the additional contour marked by B in Fig. 2(b). This additional pass goes along the left side of the imaginary axis from the point $\sin\varphi$ down to the point n and then back along the right side of the imaginary axis. Analytically this pass can be described using the function ξ_- defined by Eq. (18) for the range of parameter β specified as $0<\beta<i\cdot\sin(\varphi-\alpha)$, where $\alpha=\arcsin(n)$. Introducing for convenience a new real parameter $\gamma=-i\beta$ the equation of the bypass can be written as

$$\xi_- = i(\sin\varphi\sqrt{1-\gamma^2} - \gamma\cos\varphi), \quad 0<\gamma<\sin(\varphi-\alpha). \tag{24}$$

Thus in the formula (21) the pass of integration C has to be replaced for the pass $C+B$ and after the integration with respect to the variable p the result for the reflected temperature field will be

$$T_R = \frac{r}{4(\pi D_1 t)^{3/2}} \left\{ \int_0^\infty \exp\left[-\frac{(\beta^2+1)r^2}{4D_1 t}\right] \mathrm{Re}[R(\xi)] d\beta - \int_0^{\sin(\varphi-\alpha)} \exp\left[-\frac{(1-\gamma^2)r^2}{4D_1 t}\right] \mathrm{Im}[R(\xi_-)] d\gamma \right\}, \tag{25}$$

for $n<1$ and $\sin\varphi>n$.

In elastodynamics, an additional term, associated with integration along the branch cut, is interpreted as head or conical waves originated from the interface. By analogy, the same name can be used in Eq. (25) for the second term in the brackets, although the understanding of its physical meaning requires a more rigorous consideration.

C. Transmitted field

To derive the expression for the transmitted field we can follow the procedure developed for the reflected field. The complication here is that the exponent in the integrand in Eq. (10) contains two radicals. Let us introduce the real parameter τ

$$-i\xi x - z\sqrt{\xi^2+n^2} + z_0\sqrt{\xi^2+1} = \tau. \tag{26}$$

The properties of this equation were investigated by Cagniard [6]. He found, that for real τ, solutions $\xi(\tau)$ can be pure imaginary or complex. The value of parameter τ_0 when the contour $\xi(\tau)$ goes out from the imaginary axis to the complex plane corresponds to the singularity of the derivative $\partial\xi/\partial\tau$. Thus we come to the equation

$$x + \frac{zu}{\sqrt{n^2-u^2}} - \frac{z_0 u}{\sqrt{1-u^2}} = 0, \tag{27}$$

where a change of variable is introduced as $\xi=iu$. According to Cagniard this equation has a simple geometrical interpretation as follows. By introducing two angles φ and θ in such a way that $u=\sin\varphi$ and $u/n=\sin\theta$, Eq. (27) can be rewritten as

$$x = z_0\tan\varphi + |z|\tan\theta, \tag{28}$$

which means that for given point $B(x,z)$ the sum of projections of segments AC and CB on the x axis equals segment OD, as shown in Fig. 1. This proves that Eq. (27) always has a real root u_0 satisfying the condition

$$\frac{\sin\varphi}{\sin\theta} = n. \tag{29}$$

In this expression we can recognize Snell's law. From above two important conclusions can be deduced. First, the Cagniard contour for transmitted field crosses the imaginary axis in the complex ξ plane at the point $\sin\varphi$ as it was for the reflected field. Second, according to Snell's law, the angle φ for the transmitted field cannot exceed the critical value $\alpha=\arcsin(n)$ and the Cagniard contour never crosses the branch cut, which starts from the smallest of the two values 1 and n. Thus, the contour of integration resembles the case shown in Fig. 2(a), although it is not a hyperbola.

The value of parameter τ_0 correspondent to the root u_0 can be found as

$$\tau_0 = R_1 + nR_2, \tag{30}$$

where $R_1=AC$, $R_2=CB$, as shown in Fig. 1, provided that point C is found to satisfy condition (29). Introducing for convenience dimensionless parameter β

$$\beta = \sqrt{\frac{\tau^2}{\tau_0^2} - 1}, \tag{31}$$

we obtain the Green function for the transmitted field in the form

$$T_P=\frac{(R_1+nR_2)^2}{4(\pi D_1 t)^{3/2}}\int_0^\infty \beta \exp\left[-\frac{(\beta^2+1)(R_1+nR_2)^2}{4D_1 t}\right]\mathrm{Re}\left(\frac{\sqrt{\xi^2+1}P(\xi)}{-ix\sqrt{\xi^2+1}\sqrt{\xi^2+n^2}-z\xi\sqrt{\xi^2+1}+z_0\xi\sqrt{\xi^2+n^2}}\right)d\beta, \quad (32)$$

where function $P(x)$ is the transmission coefficient defined by expression (12), and ξ is a function of β defined by Eqs. (26) and (31).

III. PROPERTIES OF SOLUTIONS

A. Particular cases

Consider the case when the lower half-space is a nonconductive medium, $k_2=0$. Then $\chi=0$, $R(\xi)=1$ and the reflected field will be

$$T_R=\frac{r}{4(\pi D_1 t)^{3/2}}\int_0^\infty \exp\left[-\frac{(\beta^2+1)r^2}{4D_1 t}\right]d\beta$$
$$=\frac{1}{4\pi D_1 t}\exp\left[-\frac{r^2}{4D_1 t}\right], \quad (33)$$

which coincides with the temperature Green function for the half-space in 2D geometry [2]. As can be seen from Eq. (2), with $k_2=0$ we obtain the vanishing solution in the second medium even if $\rho_2 c_2\neq 0$. Thus to obtain the correct limit $T_2\rightarrow 0$ for the transmitted field in the final solution, we have to put $D_2=0$, $1/n=0$. In vacuum, we assume the limit $D_2\rightarrow 0$.

Simple formulas can be derived in the short time approximation for $x=0$. For the reflected field we will have $\xi=\beta$. If the reflection coefficient $R(\beta)$ in the integrand in Eq. (23) changes slowly in comparison with the exponent, we can bring the value $R(0)$ out of the integral sign and integrate the rest, obtaining

$$T_R=\frac{1}{4\pi D_1 t}\frac{1-\chi n}{1+\chi n}\exp\left[-\frac{(z+z_0)^2}{4D_1 t}\right], \quad z>0. \quad (34)$$

Under the same assumption that the exponential factor gives the main contribution to the integral, for the transmitted field we obtain

$$T_P=\frac{1}{4\pi D_2 t}\frac{2}{n(1+\chi n)}\sqrt{\frac{z-z_0/n}{z-z_0 n}}\exp\left[-\frac{(z-z_0/n)^2}{4D_2 t}\right],$$
$$z<0. \quad (35)$$

The range of validity of expressions (34) and (35) can be roughly evaluated as $(z+z_0)^2/(4D_1 t)\gg 1$, but it depends also on the values of n and χ.

B. Hadamard descent

It is known that by integrating over one spatial variable the Green function for point heat source in the uniform 3D space reduces to the Green function in 2D space and so on. This is a particular case of so called Hadamard's method of descent. For the 2D problem with an interface considered in this paper, it can be seen, that after integration over the x variable in the infinite limits the equations (1), (2), and boundary conditions (3) reduce to the 1D equations and boundary conditions. The same should be true for the solutions derived and this can be a test of their validity.

By elementary transformations it can be shown, that integration of the Green's functions (23) or (25) along the x axis gives

$$\int_{-\infty}^{+\infty}T_R\,dx=\frac{1-\chi n}{1+\chi n}\frac{1}{2\sqrt{\pi D_1 t}}\exp\left[-\frac{(z+z_0)^2}{4D_1 t}\right] \quad (36)$$

for both cases $n>1$ and $n<1$. The right hand side expression is the Green's function for the reflected field of an instantaneous point source in 1D geometry [2], as anticipated.

The same test can be done for the transmitted field (32), yielding

$$\int_{-\infty}^{+\infty}T_P\,dx=\frac{2}{n(1+\chi n)}\frac{1}{2\sqrt{\pi D_1 t}}\exp\left[-\frac{(z-z_0/n)^2}{4D_2 t}\right], \quad (37)$$

that is the Green function for the transmitted field in 1D case [2]. One-dimensional Green's function (37) is usually interpreted as a field generated by an image source located at the point z_0/n. In contrast, as can be seen from Eq. (32), there is no point image source for the 2D transmitted field.

C. Numerical computation

Figure 3 illustrates the numerical computations of the reflected field in the vicinity of the interface for the set of

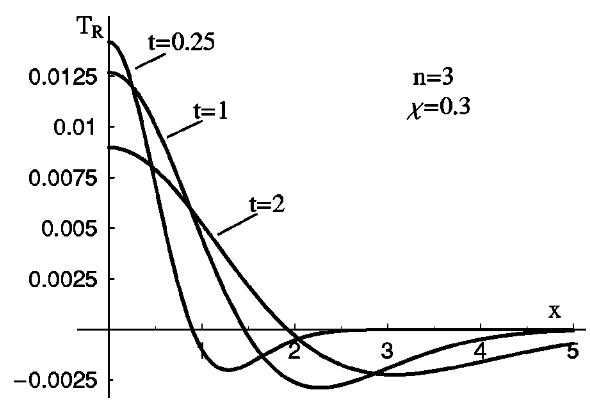

FIG. 3. Reflected field T_R in the vicinity of interface calculated by formula (23), when the point source is located at a distance $z_0=1$ cm from the interface for various times t: 0.25, 1, 2 s. Distance along the x axis is in cm, thermal diffusivity of the upper half-space is chosen 1 cm^2/s, relative values of thermal parameters of the two media are $n=3$, $\chi=0.3$.

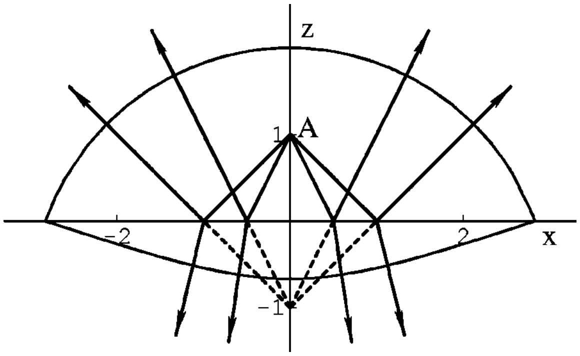

FIG. 4. Rays (arrowhead lines) and wave surfaces of reflected and transmitted temperature fields for $n=3$. The point source is located at $z_0=1$ cm.

parameters $n=3$, $\chi=0.3$. The parameters were chosen in such a way that the coefficient of reflection in 1D case [see Eq. (36)] is positive: $(1-\chi n)/(1+\chi n)>0$. As can be seen from Fig. 3, the reflected field in 2D geometry changes sign along the interface. It is connected with the fact that in 2D case the boundary conditions (3) impose relations not only on the heat flux perpendicular to the interface, but also on the heat flux along the interface.

IV. RAYS AND STREAMLINES

In the preceding section Snell's law for the rays of the temperature field was obtained. So far it does not make much sense because the definition of rays was not given. To do that, in analogy with optics [10] we can introduce the thermal distance between the point source A and the point of observation B as

$$\int_A^B \frac{ds}{\sqrt{D}}, \tag{38}$$

where ds is an element of length. Then the ray between the points A and B can be defined as a path correspondent to the shortest thermal distance. The shortest means a local minimum in comparison with all other curves in the nearest neighborhood. With this analog of Fermat's principle the Snell's law in the form (29) can be deduced. For the case when the point source A and the observation point B lay at the different sides from the interface, the thermal distance between points A and B taken along ray will be

$$\frac{R_1}{\sqrt{D_1}}+\frac{R_2}{\sqrt{D_2}}=\frac{1}{\sqrt{D_1}}(R_1+nR_2), \tag{39}$$

which is consistent with Eq. (30).

A curve, comprised of points, which lay at the same thermal distance taken along rays from the point heat source, can be called a wave surface. It can be shown that for the transmitted field T_P, the rays constructed as described above are orthogonal to the wave surfaces $R_1+nR_2=\text{const}$. A simulated diagram of rays and wave surfaces for reflected and transmitted fields for $n=3$ is shown in Fig. 4.

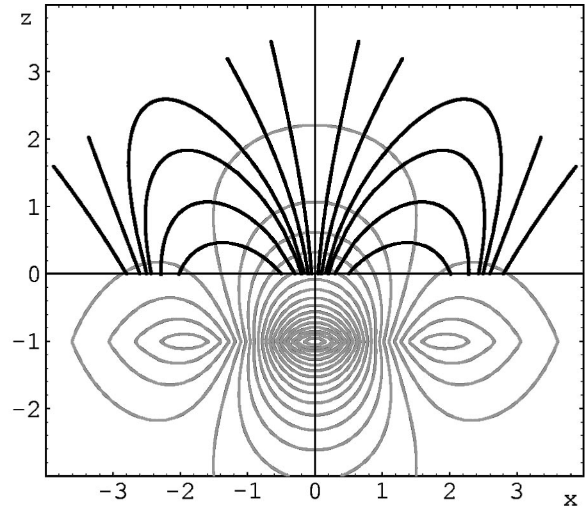

FIG. 5. Streamlines (black) and contour lines (gray) for the reflected field for $n=3$, $\chi=0.3$, $D_1=1$ cm^2/s, $t=1$ s. The point source is located at $z_0=1$ cm. The contour lines of reflected field have physical meaning only for $z>0$.

It should be noted that definitions of rays and wave surfaces do not contain dependence on time, provided the heat source is motionless and the properties of medium do not change with time. In this case the rays and wave surfaces are stationary and can be constructed in the whole space independently of time. This is consistent with the fact that the temperature field as described by the classical heat conduction equation appears simultaneously in the whole space after switching on the heat source.

An important difference of thermal rays from optical rays is that they do not represent the directions of energy flux in the medium. According to the Fourier law, the heat flux $\vec{Q}$ is proportional to the temperature gradient $\vec{\nabla}T$:

$$\vec{Q}=-k\vec{\nabla}T. \tag{40}$$

Thus, to see the directions of heat flow, one has to plot gradient lines of the temperature field, otherwise called streamlines. By definition, the streamline is a curve whose tangent at each point coincides with the direction of gradient of the field at that point. The gradient lines are orthogonal to the lines of constant temperature—isotherms. The example of gradient lines for the reflected temperature field along with its contour lines (isotherms) is shown in Fig. 5. From comparison of Figs. 4 and 5 it is clear that for the reflected field the gradient lines of heat flux do not coincide with rays. This illustrates the principal difference between diffusive and hyperbolic waves.

Both approaches for representation of the temperature field—rays and streamlines—can serve different purposes. Rays can be useful for calculation of temperature field in the presence of curved boundaries, in particular for thermal lens design, as was earlier suggested by Burt [11]. The map of streamlines can be helpful for understanding the heat exchange between layers in measurements of thermal properties of composite structures.

V. CONCLUSIONS

We have derived the exact analytical solutions for the Green functions of the temperature field created by a line instantaneous heat source in a medium with an interface. The derivation was based on the application of the Cagniard–de Hoop method originally developed for elastic wave propagation. The solutions obtained in this way have clear mathematical structure and permit of physical interpretation.

The Green's functions derived for 2D geometry help to reveal phenomena that cannot be simulated by 1D models. Thus, 2D solutions allow us to observe the change of sign of the reflected temperature field along the interface as shown in Fig. 3 and the spatial configuration of isotherms and streamlines as shown in Fig. 5.

The temperature Green functions obtained by the Cagniard–de Hoop method exhibit some wavelike features. The laws of reflection and refraction for the rays of the temperature field following from these solutions coincide with those for the plane thermal waves. From the other side, the rays of temperature field, formally introduced via the variational principle in analogy with optical rays, do not possess all properties of true rays. While the thermal rays obey the laws of reflection and refraction, they do not represent the directions of heat flux in the medium, as can be seen by comparison of Figs. 4 and 5. In fact, temperature rays simply provide us a convenient way of calculation of temperature fields, as do fictitious heat sources in the method of images.

In should be noted that the ray concept was involved for the physical interpretation of solutions and was not used during the derivation of the Green functions. In this sense it can be thought of as unnecessary. Nevertheless, thinking in terms of rays can suggest some useful analogies and hypotheses. At the same time, the exact solutions help us to comprehend the limits of wave analogies for temperature fields and thus to avoid some common misinterpretations.

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